Dog Rabbit Pursuit
Simplest Case
Differential Equation Solution
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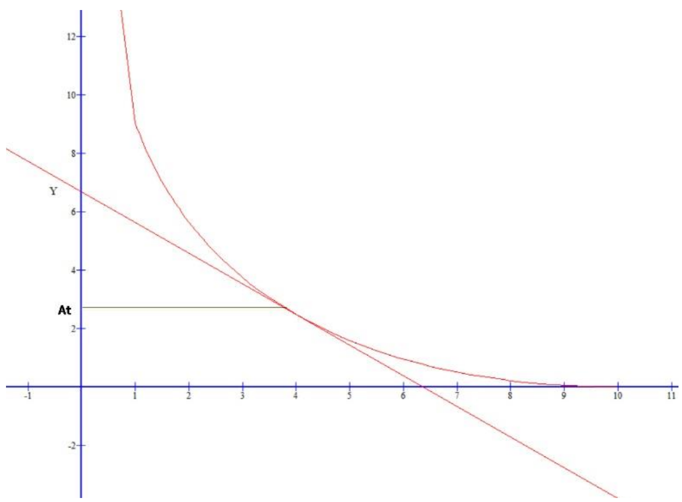
While researching this topic, I came across several different sources that were either too detailed for me (using all sorts of vector terminology) or omitted statements that would clarify the method used. I decided to put down all of the information that I found for a simple pursuit problem.

For this example, a dog pursues a rabbit. Both the dog and rabbit are moving at the same constant velocity A. The rabbit is moving directly up the Y axis, and the dog is always running directly towards the rabbit. The dog starts at the point <L,0>. The Rabbit starts at <0,0>. In this drawing, L=10, but we will keep it general.

Here is a graph showing the situation at the moment the rabbit is at t seconds elapsed from the start of the chase.



From the graph, you see that we have defined the slope of the dog's path in terms of x and y. (and time t as well) Our first goal is to find an equation for y as a function of x. Let's start with the slope and see where it leads.

$$\frac{dy}{dx} = \frac{y - At}{x}$$

This gives us an equation with y and x, but also includes the time variable t. We need to get that in terms of x and y. Rearranging:

$$x\frac{dy}{dx} - y = -At$$

Take the derivative of both sides. Remember that d(uv) = udv + vdu for the first term. Set u=x, v=y'. So du=dx and dv=y''.

$$xy'' = -A\frac{dt}{dx}$$

We now try to define dt/dx in terms x and y. This will give us a differential equation, which we hope to solve to get our equation for y as a function of x.

Let's define S as the arc length. We will state the equation for arc length here as a derivative which you would integrate to get the arc length. (*See Appendix A to see how this was derived*.)

$$ds = \sqrt{1+(y')^2}dx$$

With this information, we know ds/dx. We also know that ds/dt is just the velocity of the dog, A (same as rabbit).

If we take the reciprocal of the velocity, we get dt/ds. This is 1/A, but we have to be careful about the interpretation. Since s is decreasing as the dog runs along and t is increasing. The slope is negative and 1/A is negative.

So now we can multiply these values to get dt/dx in terms of x and y.

$$\frac{dt}{ds}\frac{ds}{dx} = \frac{dt}{dx}$$

$$-\frac{1}{A} * \sqrt{1+(y')^2} = \frac{dt}{dx}$$

Going back to our earlier equation:

$$xy'' = -A\frac{dt}{dx}$$

We can now substitute in the value of dt/dx:

$$xy'' = \sqrt{1+(y')^2}$$

Now we can solve this differential equation to determine y as a function of x.

Let's simplify by letting p=y'. So, p'=y''. Our equation looks like this:

$$xp' = \sqrt{1+p^2}$$

Rearranging:

$$\frac{dp}{\sqrt{1+p^2}} = \frac{dx}{x}$$

Integrating both sides: *(See Appendix B for how this was done)*

$$\ln\left(\sqrt{1+p^2}+p\right) + c_1 = \ln(x) + c_2$$

We need to account for the constants, and simplify. Both constants can be combined into one on the right side of the equation. Since this is a constant, we can also define another constant that is the logarithm of this constant. We do this so that we can combine the right side of the equation into one logarithmic term. That way, we can remove the logarithm from both sides of the equation.

First, we combine, c3 = c2-c1. Then we let c3=-ln(c). So, our equation becomes:

$$\ln\left(\sqrt{1+p^2}+p\right) = \ln(x) - \ln(c)$$

We can use logarithm rules to change the subtracting logarithms into a quotient, and then remove the logarithm from each side.

$$\sqrt{1+p^2} = \frac{x}{c} - p$$

$$1 + p^2 = \left(\frac{x}{c}\right)^2 + p^2 - 2\left(\frac{px}{c}\right)$$

$$2px = \frac{x^2}{c} - c$$

$$p = \frac{1}{2}\left(\frac{x}{c} - \frac{c}{x}\right) = \frac{dy}{dx}$$

Looking at the starting point of the dog, you can see the slope is 0, so dy/dx at that point is 0. We can use this to determine the constant c.

$$0 = \left(\frac{x}{c} - \frac{c}{x}\right) = x^2 - c^2$$

$$x = c$$

At this point, x=L, so we can replace the constant c with L.

$$p = \frac{1}{2}\left(\frac{x}{L} - \frac{L}{x}\right) = \frac{dy}{dx}$$

So, now we have an equation for dy/dx. If we integrate, we can get what we need…and equation for y as a function of x.

$$y = \int \frac{1}{2}\left(\frac{x}{L} - \frac{L}{x}\right) dx = \frac{x^2}{4L} - \frac{L\ ln(x)}{2} + constant$$

You know that y=0 when x=L. Substituting this in, you can solve for the constant.

$$constant = \frac{L\ ln(L)}{2} - \frac{L^2}{4L}$$

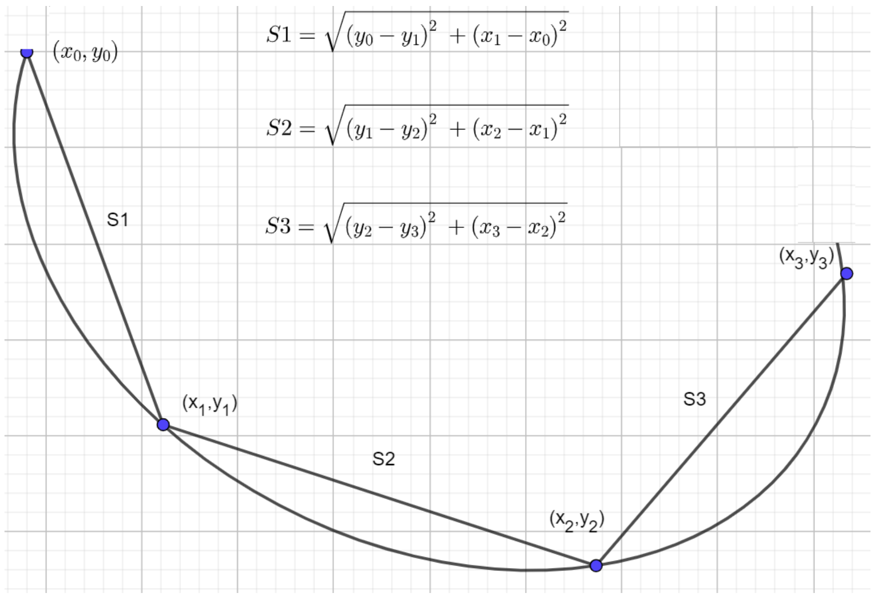
$$y = \frac{x^2}{4L} - \frac{L\ ln(x)}{2} + \frac{L\ ln(L)}{2} - \frac{L^2}{4L} = \frac{x^2 - L^2}{4L} - \frac{L}{2}(\ln(x) - \ln(L))$$

The solution to our problem is:

$$y = \frac{x^2 - L^2}{4L} - \frac{L}{2}\ln\left(\frac{x}{L}\right)$$

Appendix A
Derivation of Curve Length Formula

To find the length of curve, draw line segments as shown. S1, S2 and S3 are determined by using the Pythagorean Theorem.



The three equations can also be structured using delta notation for each of the subtractions. For example.

$$S1 = \sqrt{\Delta x_1{}^2 + \Delta y_1{}^2}$$

At this point, a division and multiplication by $\Delta x_1{}^2$ will lead to this:

$$S1 = \Delta x_1 \sqrt{\frac{\Delta x_1{}^2}{\Delta x_1{}^2} + \frac{\Delta y_1{}^2}{\Delta x_1{}^2}}$$

The approximate curve length is the sum of all of these: S1+S2+S3. As the line segments get smaller and smaller, adding more and more segments will get you closer to the real curve length.

$$dS = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

An integration using the end points of the curve as limits will yield the exact length of the curve. A similar equation can be derived using dy.

Appendix B

Integration of

$$\frac{dp}{\sqrt{1+p^2}} = \frac{dx}{x}$$

Starting with the more difficult term on the left:

$$\int \frac{1}{\sqrt{1+p^2}} dp$$

The denominator should remind you of trigonometric functions. If this were a subtraction rather than an addition, you might try the substitution based on $sin^2 + cos^2 = 1$. Since this is an addition, we consider a related suggestion.

$$\frac{sin^2}{cos^2} + \frac{cos^2}{cos^2} = tan^2 + 1 = \frac{1}{cos^2} = sec^2$$

Given this, let's try substituting $p = tan(u)$. Therefore, $dp = sec^2(u)du$.

Now,

$$\sqrt{1+p^2} = \sqrt{tan^2(u)+1} = \sec(u)$$

And

$$u = tan^{-1}(p)$$

$$\int \frac{1}{\sqrt{1+p^2}} dp = \int \frac{1}{\sec(u)} sec^2(u)du = \int \sec(u)du$$

Before going any further, let's look at our original equality:

$$\frac{dp}{\sqrt{1+p^2}} = \frac{dx}{x}$$

The right-hand side, when integrated results in a natural log term, so perhaps we should look into how to express the integral on the left as something logarithmic, where a derivative is divided by the function that produces it.

We are working with a Secant term. Let's look at what the derivative looks like and try to form a strategy.

$$d\sec(u) = \sec(u)\tan(u)$$

$$d\tan(u) = sec^2(u)$$

If we add these two together:

$$d\,(\sec(u) + \tan(u)) = sec^2(u) + \sec(u)\tan(u)$$

Back to our solution so far, let's multiply and divide by $\sec(u) + \tan(u)$

$$\int \frac{1}{\sqrt{1+p^2}} dp = \int \frac{1}{\sec(u)} sec^2(u)du = \int \sec(u)du = \int \frac{sec^2(u) + \tan(u)\sec(u)}{\sec(u) + \tan(u)} du$$

Now, if we let $s = \tan(u) + \sec(u)$ and $ds = sec^2(u) + \sec(u)\tan(u)$, our operation becomes:

$$\int \frac{ds}{s} = \log(s) + c$$

Substitute back for s:

$$\int \frac{ds}{s} = \log(\tan(u) + \sec(u)) + c$$

Substitute back for p: ($u = tan^{-1}(p)$)

$$\frac{dp}{\sqrt{1+p^2}} = \log(\tan(tan^{-1}(p)) + \sec(tan^{-1}(p))) + c$$

In a unit circle, we know that $sin^2x + cos^2x = 1, so\ tan^2x + 1 = sec^2x$.

This means that

$$\sec(x) = \sqrt{tan^2x + 1}$$

Let $p = tan(x)\ so\ x = tan^{-1}(p)$. This leads to:

$$\sec\left(tan^{-1}(p)\right) = \sqrt{p^2 + 1}$$

So now we can substitute back into our solution:

$$\frac{dp}{\sqrt{1+p^2}} = \log\left(p + \sqrt{p^2+1}\right) + c$$

And then back to our original premise:

$$\frac{dp}{\sqrt{1+p^2}} = \frac{dx}{x}$$

$$\log(p + \sec(tan^{-1}(p))) + c_1 = \log(x) + c_2$$

So:

$$\log(p + \sqrt{p^2+1}) + c_1 = \log(x) + c_2$$